§. Muckenhoupt A_p Weights

→ A **weight** (on $\mathbb{R}^n$) is a locally integrable, a.e. positive function $w(x)$ on $\mathbb{R}^n$.

Remark: A weight w can only be 0 or ∞ on a set of Lebesgue measure zero

⇒ the same is true automatically of $\frac{1}{w}$.

⇒ If w is a weight on $\mathbb{R}^n$ and $\frac{1}{w}$ is locally integrable, then $\frac{1}{w}$ is also a weight.

→ Given a weight w, we denote by: $\boxed{w(E) := \int_E w(x)dx}$ the w-measure of any measurable set E. Since weights are locally integrable, $w(E) < \infty$ if E is a bounded set.

→ The weighted L^p-spaces associated with any weight w: $\boxed{L^p(\mathbb{R}^n; dw)}$ or simply $\boxed{L^p(w)}$

$$\boxed{\begin{aligned} \|f\|^p_{L^p(w)} &:= \int_{\mathbb{R}^n} |f(x)|^p dw(x) \\ &= \int_{\mathbb{R}^n} |f(x)|^p \underbrace{w(x)dx}_{=:dw} \end{aligned}} \quad \forall 1<p<\infty$$

→ Recall the (uncentered) Hardy-Littlewood maximal operator on $\mathbb{R}^n$ (over cubes):

$$\boxed{Mf(x) := \sup_{Q \ni x} \langle |f| \rangle_Q}$$

where the supremum is over all cubes Q (w/ sides parallel to the axes) that contain x.

→ Classical result: $\forall 1<p<\infty$, $\exists$ constant $C_p(n)>0$ such that $\|Mf\|_{L^p} \leq C_p(n)\|f\|_{L^p}$, $\forall f \in L^p(\mathbb{R}^n)$.

→ Fairly easy extension: the same holds for any doubling measure μ on $\mathbb{R}^n$

$(\exists C>0 \text{ s.t. } \mu(2B(x;r)) \leq C\mu(B(x;r)), \forall x \in \mathbb{R}^n, r>0)$

→ Focus now on measures that are absolutely continuous wrt Lebesgue measure, i.e. measures of the form $d\mu(x) = w(x)dx$ for some weight $w(x)$.

→ **QUESTION**: Is there a characterization of all weights $w(x)$ such that $\|Mf\|_{L^p(w)} \leq C\|f\|_{L^p(w)}$ holds for all $f \in L^p(w)$? $(1<p<\infty)$

→ **Answer**: We show that any such weight w must satisfy the rather strange-looking condition:

$$\boxed{\sup_Q \left(\langle w \rangle_Q \left\langle w^{-\frac{1}{p-1}} \right\rangle_Q^{p-1} \right) \leq C^p}$$

(will be referred to as the A_p condition).

Remarkable fact: The converse also holds, i.e. the A_p condition implies boundedness of M on $L^p(w)$.

$M: L^p(w) \to L^p(w)$ is bounded $\Rightarrow$ A_p condition for w

Suppose that $\boxed{\|Mf\|_{L^p(w)} \leq C\|f\|_{L^p(w)},\ \forall f \in L^p(w)}$ holds for some weight w.

Apply to $f\mathbb{1}_Q$ for some cube Q:

$$M(f\mathbb{1}_Q)(x) = \sup_{R \ni x} \langle |f|\mathbb{1}_Q\rangle_R \geq \langle |f|\mathbb{1}_Q\rangle_Q = \langle |f|\rangle_Q,\ \forall x \in Q$$

$$\Rightarrow \int_Q M(f\mathbb{1}_Q)^p\, dw \geq \int_Q \langle |f|\rangle_Q^p\, dw = w(Q)\langle |f|\rangle_Q^p$$

$$\Rightarrow \langle |f|\rangle_Q^p \leq \frac{1}{w(Q)}\int_Q M(f\mathbb{1}_Q)^p\, dw \leq \frac{1}{w(Q)}\int_{\mathbb{R}^n} M(f\mathbb{1}_Q)^p\, dw \leq \frac{C^p}{w(Q)}\int_{\mathbb{R}^n} (|f|\mathbb{1}_Q)^p\, dw$$

$$\Rightarrow \boxed{\langle |f|\rangle_Q^p \leq \frac{C^p}{w(Q)}\int_Q |f|^p\, dw},\ \forall f \in L^p(w),\ \forall \text{ cube } Q.$$

→ Quick look at $p=2$: $\langle |f|\rangle_Q^2 \leq \dfrac{C^2}{w(Q)}\displaystyle\int_Q |f|^2\, dw$

$$\frac{1}{|Q|^2}\left(\int_Q |f(x)|\, dx\right)^2 \leq \frac{C^2}{w(Q)}\int_Q |f(x)|^2 w(x)\, dx$$

What if we choose f so that the integrands are the same? $|f| = |f|^2 w \Rightarrow |f| = w^{-1}$

$\Rightarrow$ Choose $f(x) = \dfrac{1}{w(x)}$ (assume locally integrable)

$$\Rightarrow \text{condition becomes}\quad \langle w^{-1}\rangle_Q^2 \leq \frac{C^2}{w(Q)}\, w^{-1}(Q)$$

$$\frac{1}{|Q|^2}\, w(Q)\, w^{-1}(Q) \leq C^2$$

$$\boxed{\sup_Q \langle w\rangle_Q \langle w^{-1}\rangle_Q \leq C}$$

The A_2 condition!

→ Generally: $\dfrac{1}{|Q|^p}\left(\displaystyle\int_Q |f|\, dx\right)^p \leq \dfrac{C^p}{w(Q)}\displaystyle\int_Q |f|^p w(x)\, dx$

$$|f| = |f|^p w \Rightarrow w^{-1} = |f|^{p-1}$$

$\Rightarrow$ Choose $\boxed{f(x) = w^{-\frac{1}{p-1}}(x)}$:

$$f^p = w^{-\frac{p}{p-1}}$$

$$f^p \cdot w = w^{1-\frac{p}{p-1}} = w^{-\frac{1}{p-1}}$$

$$\frac{1}{|Q|^p}\left(\int_Q w^{-\frac{1}{p-1}}\, dx\right)^p \leq \frac{C^p}{w(Q)}\left(\int_Q w^{-\frac{1}{p-1}}\, dx\right)$$

$$\frac{1}{|Q|^p}\, w(Q)\left(\int_Q w^{-\frac{1}{p-1}}\right)^{p-1} \leq C^p,\ \forall \text{ cube } Q$$

$$\boxed{\sup_Q \left(\langle w\rangle_Q \langle w^{-\frac{1}{p-1}}\rangle_Q^{p-1}\right) \leq C}$$

The A_p Condition
$(1 < p < \infty)$.

Def.: We say w is an **A_p weight** $(1<p<\infty)$ if:

$$\boxed{[w]_{A_p} := \sup_Q \langle w\rangle_Q \langle w^{-\frac{1}{p-1}}\rangle_Q^{p-1}} < \infty$$

where supremum is over all cubes in $\mathbb{R}^n$ (w/ sides parallel to the axes).

Remark: dyadic A_p weights $\boxed{A_p^{\mathcal{D}}}$: same def., only sup over $Q\in\mathcal{D}$.

Remark: If $w\in A_p$ $(1<p<\infty)$ then the weight $w' := w^{-\frac{1}{p-1}}$ is in the class $A_{p'}$ (where p, p' are Hölder conjugates)

$$\frac{1}{p}+\frac{1}{p'}=1 \qquad \frac{1}{p'}=\frac{p-1}{p} \qquad p'=\frac{p}{p-1} \qquad p+p'=pp' \qquad (p-1)(p'-1)=1$$

$$\langle w'\rangle_Q \langle (w')^{-\frac{1}{p'-1}}\rangle_Q^{p'-1} = \langle w^{-\frac{1}{p-1}}\rangle_Q \langle w\rangle_Q^{p'-1} = \langle w^{-\frac{1}{p-1}}\rangle_Q^{(p-1)(p'-1)} \langle w\rangle_Q^{p'-1}$$

$$(w')^{-\frac{1}{p'-1}} = w^{\frac{1}{(p-1)(p'-1)}} = w \qquad = \left(\langle w\rangle_Q \langle w^{-\frac{1}{p-1}}\rangle_Q^{p-1}\right)^{p'-1} \le [w]_{A_p}^{p'-1}$$

$$\Rightarrow w'\in A_{p'} \text{ with } \boxed{[w']_{A_{p'}} = [w]_{A_p}^{p'-1}}$$

Remark: Often used in duality, to preserve Lebesgue inner product:

$w\in A_p$: $(L^p(w))^* = L^{p'}(w)$, i.e. $\forall\ \Lambda\in(L^p(w))^*$, $\exists!\ g\in L^{p'}(w)$ s.t. $\Lambda f = (f,g)_w = \int_{\mathbb{R}^n} fg\,dw$ $\forall f\in L^p(w)$.

Sometimes it's useful to preserve $(f,g)_{dx}$ (esp. in dyadic, Haar settings):

$$g\in L^{p'}(w) \Rightarrow \|g\|_{L^{p'}(w)}^{p'} = \int |g|^{p'}dw = \int |g|^{p'} w^{p'} \underbrace{w^{1-p'}}_{=w^{-\frac{1}{p-1}}} = \int |gw|^{p'}\, \underline{\underline{w}}' = \|gw\|_{L^{p'}(w')}^{p'}$$

$$g\in L^{p'}(w) \Leftrightarrow (gw)\in L^{p'}(w')$$

So $\Lambda f = (f,g)_w = \int f\underbrace{gw}_{\in L^{p'}(w')}$ can be thought of as $\Lambda f = \int fg'$, with $g'\in L^{p'}(w')$

$$\boxed{(L^p(w))^* \simeq L^{p'}(w'), \text{ i.e. } \forall \Lambda\in(L^p(w))^*, \exists! g\in L^{p'}(w') \text{ s.t. } \Lambda f = (f,g) = \int fg\,dx}$$

Useful in bounding operators on weighted spaces:

NTS:

$$\|Tf\|_{L^p(w)} \lesssim \|f\|_{L^p(w)} \Leftrightarrow \sup_{g\in L^{p'}(w')} |(Tf,g)| \lesssim \|f\|_{L^p(w)} \|g\|_{L^{p'}(w')}.$$